12.4. LOCALIZATION OF TOEPLITZ OPERATORS*

Let H^2 and H^∞ denote the Hardy subspaces of $L^2(\mathbb{T})$ and $L^\infty(\mathbb{T})$, respectively, consisting of the functions with zero negative Fourier coefficients, and let P be the orthogonal projection from $L^2(\mathbb{T})$ onto H^2. For φ in $L^2(\mathbb{T})$ the Toeplitz operator with symbol φ is defined on H^2 by $T_\varphi f = P(\varphi f)$. Much of the interest in Toeplitz operators has been directed toward their spectral characteristics either singly or in terms of the algebras of operators which they generate. In particular, one seeks conceptual determinations of why an operator is or is not invertible and more generally Fredholm. One fact which one seeks to explain is the result due to Widom [1] that the spectrum $\sigma(T_\varphi)$ of an arbitrary Toeplitz operator is a connected subset of $\mathbb{C}$ and even [2] the essential spectrum $\sigma_e(T_\varphi)$ is connected. The latter result implies the former in view of Coburn's lemma.

An important tool introduced in [2, 3] is the algebraic notion of localization. Let $\mathfrak{J}$ denote the closed algebra generated by all Toeplitz operators and QC be the subalgebra $\{H^\infty + C\} \cap \overline{\{H^\infty + C\}}$ of L^∞, where C denotes the algebra of continuous functions on $\mathbb{T}$. Each ξ in the maximal ideal space M_{QC} of QC determines a closed subset X_ξ of M_{L^∞}, and one can show that the closed ideal $\mathfrak{V}_\xi$ in $\mathfrak{J}$ generated by $\{T_\varphi : \hat{\varphi}|_{X_\xi} \equiv 0\}$ is proper and that the local Toeplitz operator $T_\varphi + \mathfrak{V}_\xi$ in $\mathfrak{J}_\xi = \mathfrak{J}/\mathfrak{V}_\xi$ depends only on $\hat{\varphi}|X_\xi$. Moreover, since $\bigcap_{\xi \in M_{QC}} \mathfrak{V}_\xi$ equals the ideal $\mathcal{K}$ of compact operators on H^2, properties which are true modulo $\mathcal{K}$ can be established "locally." For example, T_φ is Fredholm if and only if $T_\varphi + \mathfrak{V}_\xi$ is invertible for each ξ in M_{QC}. These localization results are established [4] by identifying QC as the center of $\mathfrak{J}/\mathcal{K}$. One unanswered problem concerning local Toeplitz operators is:

<u>Conjecture 1.</u> The spectrum of a local Toeplitz operator is connected.

In [4] it was shown that many of the results known for Toeplitz operators have analogues valid for local Toeplitz operators. Unfortunately a proof of the connectedness would seem to require more refined knowledge of the behavior of H^∞ functions of M_{H^∞} than available and the result would imply the connectedness of $\sigma_e(T_\varphi)$.

A more refined localization has been obtained by Axler replacing X_ξ by the subsets of M_{L^∞} of maximal antisymmetry for $H^\infty + C$ using the fact that the local algebras $\mathfrak{J}_\xi$ have nontrivial centers and iterating this transfinitely.

There is evidence to believe that the ultimate localization should be to the closed support X_η in M_{L^∞} for the representing measure μ_η for a point η in M_{H^∞}. In particular, one would like to show that if $H^2(\mu_\eta)$ denotes the closure in $L^2(\mu_\eta)$ of the functions $\hat{\varphi}|X_\eta$ for φ in H^∞, P_η the orthogonal projection from $L^2(\mu_\eta)$ onto $H^2(\mu_\eta)$, then the map $T_\varphi \to T_{\hat{\varphi}|X_\eta}$ extends to the corresponding algebras, where the local Toeplitz operator is defined by $T_{\hat{\varphi}|X_\eta} f = P_\eta(\hat{\varphi}|X_\eta) f$ for f in $H^2(\mu_\eta)$. If η is a point in M_{L^∞}, then $H^2(\mu_\eta) = \mathbb{C}$ and it is a special case of the result [2] that $\mathfrak{J}$ modulo its commutator ideal is isometrically isometric to L^∞, that the map extends to a character in this case. A generalized spectral inclusion theorem also provides evidence for the existence of this mapping in all cases.

One approach to establishing the existence of this map is to try to exhibit the state on $\mathfrak{J}$ which this "representation" would determine. One property that such a state would have is that it would be multiplicative on the Toeplitz operators with symbols in H^∞. Call such states *analytically multiplicative*. Two problems connected with such states seem interesting.

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Conjecture 2 (Generalized Gleason—Whitney). If σ_1 and σ_2 are analytically multiplicative states on $\mathfrak{J}$ which agree on H^∞ and such that the kernels of the two representations defined by σ_1 and σ_2 are equal, then the representations are equivalent.

Conjecture 3 (Generalized Corona). In the collection of analytically multiplicative states the ones which correspond to points of $\mathbb{D}$ are dense.

One consequence of a localization to X_η, when η is an analytic disk $\mathbb{D}_\eta$, would be the following. It is possible for φ in L^∞ that its harmonic extension $\hat{\varphi}|\mathbb{D}_\eta$ agrees with the harmonic extension of a function continuous on $\mathbb{T}$. (Note that this is not the same as saying that $\hat{\varphi}$ is continuous on the boundary of $\mathbb{D}_\eta$ as a subset of M_{H^∞} which is of course always the case.) In that case the invertibility of the local Toeplitz operator would depend on a "winding number" which should yield a subtle necessary condition for T_φ to be Fredholm. Ultimately it may be that there are enough analytic discs in M_{H^∞} on which the harmonic extension $\hat{\varphi}$ is "nice" to determine whether or not T_φ is Fredholm, but that would require knowing a lot more about M_{H^∞} than we do now.

LITERATURE CITED

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